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On isochronous cases of the Cherkas system and Jacobi's last multiplier*

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Abstract

We consider a large class of polynomial planar differential equations proposed by Cherkas (1976 *Differensial'nye Uravneniya* **12** 201–6), and show that these systems admit a Lagrangian description via the Jacobi last multiplier (JLM). It is shown how the potential term can be mapped either to a linear harmonic oscillator potential or into an isotonic potential for specific values of the coefficients of the polynomials. This enables the identification of the specific cases of isochronous motion without making use of the computational procedure suggested by Hill *et al* (2007 *Nonlinear Anal.: Theor. Methods Appl.* **67** 52–69), based on the Pleshkan algorithm. Finally, we obtain a Lagrangian description and perform a similar analysis for a cubic system to illustrate the applicability of this procedure based on Jacobi's last multiplier.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

The identification of isochronous dynamical systems, whose motions are periodic in their phase space, has attracted the attention of several researchers in both the physics and mathematics communities. Special mention must be made of the recent results of Calogero *et al* [6–8]. In [10] the authors have shown that although the space of the isochronous potentials is fairly large, up to a shift $x \to x + a$ and the addition of a constant, all *rational isochronous potentials* are described by either the linear harmonic oscillator potential $V(x) = \frac{1}{2}w^2x^2$ or the isotonic

* Dedicated to Professor Francesco Calogero on his 75th birthday with great respect and admiration.

potential $V(x) = \frac{1}{8}w^2x^2 + \frac{c^2}{x^2}$, where c is a nonzero constant [1–5]. The systems described by them are characterized by a family of oscillatory solutions with the same time period $T = 2\pi/w$. However, there exist other classes of isochronous systems described by irrational potentials, for instance potentials for which the second derivative has a discontinuity.

The study of the isochronicity problem for parametric families of ordinary differential equations (ODEs) is a nontrivial problem and requires tremendous computation work. Often the high degree of computations that are generally involved tend to suppress some of the geometrical aspects of the theory. Hence it is desirable to study this problem from a different angle. In this paper, we re-examine the polynomial Cherkas system, studied in [15], from the Lagrangian dynamics point of view. We construct a Lagrangian for the Cherkas system using the Jacobi last multiplier (JLM) [18–22] and derive the corresponding Hamiltonian function. By constructing an obvious transformation of variables we can easily map the Hamiltonian to that of the linear harmonic oscillator or to an isotonic potential. This enables us to invoke another class of isochronous potentials apart from the harmonic oscillator. Tacitly we exploit the criteria of isochronicity due to Urabe [25]. In fact, in an interesting paper [9] the authors obtained, by an analysis of an Urabe-function, certain necessary and sufficient conditions for isochronicity of cubic systems reducible to a Liénard-type system. A simple proof of the criterion based on a formula from Landau and Lifshitz [16] is obtained in [23]. It turns out that in most cases the transformation is canonical. Our approach based on the Jacobi last multiplier is marked by a remarkable degree of simplicity.

In [14] the author studied the second class of Liénard system (Liénard II)

$$x'' + f(x)x'^2 + g(x) = 0,$$

with a center at the origin 0 and investigated conditions under which it exhibited isochronicity. In addition, he has given a necessary and sufficient condition for the isochronicity of 0 when f and g are analytic (not necessarily odd). This approach enables him to present in an algorithmic way the conditions for a point of Liénard II to be an isochronous center. In particular he has found, in an alternative manner, the isochrones of the quadratic Loud system [17]. Interestingly, Chouikha also classified a five-parameter family of reversible cubic systems having isochronous centers at the origin. In this paper, we study this class of cubic systems using the JLM approach. In particular, by using the JLM, we obtain a Lagrangian description of the cubic system studied in [14]. Our calculations illustrate the utility of Jacobi's last multiplier for studying the isochronicity problem of parametric families of ODEs, without having to take recourse to high power computations.

The paper is organized as follows. In section 2, we present a brief description of the Cherkas system and Jacobi's last multiplier. We also describe the connection between the JLM and Lagrangian in this section. In section 3, we outline the mapping of the potential to the harmonic oscillator and to the isotonic oscillator potentials. In section 4, we obtain specific isochronous cases of the restricted Cherkas system. In section 5, we illustrate the applicability of our procedure to a cubic system and identify certain isochronous cases which were previously obtained by Chouikha in [14], using an entirely different approach. Finally, we present a modest outlook in section 6.

2. The Cherkas system revisited

The following polynomial system:

$$\begin{aligned} \dot{x} &= y(1+x) \\ \dot{y} &= -x - a_1x^2 - a_2x^3 - a_3x^4 - a_4(x + a_5x^2)y - a_6y^2, \end{aligned} \tag{2.1}$$

where the a_i, $1 \leqslant i \leqslant 6$, are the arbitrary real coefficients, was investigated originally by Cherkas [13]. In this paper, we will consider a special case of such a system when the coefficient $a_4 = 0$, so that we can essentially restrict our analysis to the following class of systems:

$$\begin{aligned} \dot{x} &= p_1(x)y, \\ \dot{y} &= q_0(x) + q_2(x)y^2. \end{aligned} \tag{2.2}$$

Such a system corresponds to the second-order (Liénard-type) equation

$$\ddot{x} + f(x)\dot{x}^2 + g(x) = 0, \tag{2.3}$$

with

$$f(x) = -\left(\frac{p_1'(x) + q_2(x)}{p_1(x)}\right), \qquad g(x) = -p_1(x)q_0(x). \tag{2.4}$$

Sabatini in [24] studied the period function of such equations and deduced a sufficient condition for the monotonicity of the period function, or for the isochronicity of the center of the equation.

Our motivation is somewhat different, since we wish to investigate the isochronous cases of the restricted Cherkas system from the perspective of the Lagrangian dynamics. Our analysis is based on the extensive use of the JLM which is introduced in the following section.

2.1. *Jacobi last multipliers and the Lagrangian*

Consider a system of first-order ordinary differential equations:

$$\dot{x}^i = X^i(t, x^1, \ldots, x^n), \qquad i = 1, \ldots, n. \tag{2.5}$$

The solutions of the system are the integral curves of the vector field Γ in $\mathbb{R}^n \approx T\mathbb{R}$. A Jacobi multiplier $M(t, x^1, \ldots, x^n)$ for such a vector field is basically an integrating factor of the system (2.5) such that

$$\operatorname{div}(MX^i) = \frac{\partial M}{\partial t} + \sum_i \frac{\partial (MX^i)}{\partial x^i} = 0,$$

which may be written as

$$\frac{\mathrm{d}\log M}{\mathrm{d}t} + \sum_i \frac{\partial X^i}{\partial x^i} = 0. \tag{2.6}$$

Clearly for a second-order ODE, $\ddot{x} = \mathcal{F}(t, x \cdot \dot{x})$, which can be expressed as the system

$$\dot{x} = v, \qquad \dot{v} = \mathcal{F}(t, x, v),$$

it follows that

$$\frac{\mathrm{d}\log M}{\mathrm{d}t} + \frac{\partial \mathcal{F}}{\partial \dot{x}} = 0. \tag{2.7}$$

The relationship between the Lagrangian and the JLM is apparent from the following consideration. The Euler–Lagrange equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x}$$

may be written as

$$\frac{\partial^2 L}{\partial t \partial \dot{x}} + \dot{x}\frac{\partial^2 L}{\partial x \partial \dot{x}} + \mathcal{F}(x, \dot{x})\frac{\partial^2 L}{\partial \dot{x}^2} = \frac{\partial L}{\partial x}, \tag{2.8}$$

where we have used the fact that $\ddot{x} = \mathcal{F}(x, \dot{x})$. Differentiating (2.8) with respect to $\dot{x}$ we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\log\left(\frac{\partial^2 L}{\partial \dot{x}^2}\right) + \frac{\partial \mathcal{F}}{\partial \dot{x}} = 0; \tag{2.9}$$

it is being assumed that $\partial^2 L/\partial \dot{x}^2 \neq 0$. This shows that $\partial^2 L/\partial \dot{x}^2$ satisfies the defining equation (2.7) for the JLM of a second-order ODE and therefore we have

$$M = \frac{\partial^2 L}{\partial \dot{x}^2}. \tag{2.10}$$

2.2. *A Lagrangian description*

We now return to (2.3). The Jacobi multiplier for this equation is given by a solution of

$$\frac{\mathrm{d}}{\mathrm{d}t}\log M + \frac{2}{p_1(x)}(p_1'(x) + q_2(x))\dot{x} = 0. \tag{2.11}$$

Its solution is given by

$$M(x) = \left(\frac{1}{p_1(x)}\exp\left[-\int^x \frac{q_2(s)}{p_1(s)}\,\mathrm{d}s\right]\right)^2 = \left(\exp\int^x f(s)\,\mathrm{d}s\right)^2, \tag{2.12}$$

and is clearly non-negative. It is necessary to point out that for (2.3) the JLM is independent of $\dot{x}$. Using (2.10) and (2.12) it is easy to derive the Lagrangian of (2.3), which has the form

$$L(x, \dot{x}) = \tfrac{1}{2}M(x)\dot{x}^2 + R(x, t)\dot{x} + S(x, t), \tag{2.13}$$

where $R(x, t)$ and $S(x, t)$ are the arbitrary functions of their respective arguments. Their explicit forms may be fixed by comparing the Euler–Lagrange equation of motion as obtained from the above Lagrangian L and (2.3). Since

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right) = M(x)\ddot{x} + M'(x)\dot{x}^2 + R_t + R_x\dot{x}$$

and

$$\frac{\partial L}{\partial x} = \frac{1}{2}M'(x)\dot{x}^2 + R_x\dot{x} + S_x,$$

the Euler–Lagrange equation yields the equation

$$M(x)\ddot{x} + \tfrac{1}{2}M'(x)\dot{x}^2 + R_t - S_x = 0.$$

By using (2.3) and after some cancellations we arrive at the following relation:

$$S_x - R_t = -M(x)g(x) = M(x)q_0 p_1. \tag{2.14}$$

Let $S(x, t) = G_t + K(x)$ and $R = G_x$ for some function $G(x,t)$. Then $S_x - R_t = K(x)$ so that from (2.14) we find that

$$K(x) = -\int^x M(s)g(s)\,\mathrm{d}s = \int^x M(s)q_0(s)p_1(s)\,\mathrm{d}s \tag{2.15}$$

and hence,

$$L(x, \dot{x}) = M(x)\frac{\dot{x}^2}{2} + \frac{\mathrm{d}G}{\mathrm{d}t} - \int^x M(s)q_0(s)p_1(s)\,\mathrm{d}s. \tag{2.16}$$

The total derivative term is inconsequential and may be discarded. In this form, the Lagrangian broadly resembles the natural form $T - V$, where T and V represent the kinetic and potential energies, respectively. Furthermore, if we apply a Legendre transformation to this Lagrangian, we arrive at the expression for the corresponding Hamiltonian of the system, namely

$$H = \frac{1}{2}\left(\frac{p}{\sqrt{M(x)}}\right)^2 + \int^x M(s)q_0(s)p_1(s)\,\mathrm{d}s. \tag{2.17}$$

Here the conjugate momentum is defined by the usual prescription

$$p = \frac{\partial L}{\partial \dot{x}} = M(x)\dot{x}, \tag{2.18}$$

where we have omitted $R(x,t)$ since it may be tucked away in the total derivative term $\mathrm{d}G/\mathrm{d}t$ which may be safely discarded. From (2.17) it is easy to see that the potential term is given by

$$V(x) = \int^x M(s)q_0(s)p_1(s)\,\mathrm{d}s. \tag{2.19}$$

3. Mapping $V(x)$ to the linear harmonic oscillator potential/isotonic potential

In this section, we investigate the conditions under which the potential $V(x)$ may be mapped to a linear harmonic oscillator potential. Suppose there exists a transformation $x \to Q = \psi(x)$ such that $V(x) \to V(Q) = \frac{1}{2}w^2Q^2$. Defining $P = p/\sqrt{M(x)}$ it is obvious that $H = \frac{1}{2}P^2 + \frac{1}{2}w^2Q^2$. If the transformation $(p,x) \to (P,Q)$ is to be a canonical transformation, then it is necessary that the Poisson bracket $\{P,Q\} = \{p,x\}$, which leads to the following condition:

$$\psi'(x) = \sqrt{M(x)} \qquad \text{and implies} \qquad \psi(x) = \left(\int^x \sqrt{M(s)}\,\mathrm{d}s + \alpha\right) := Q. \tag{3.1}$$

Here α is a constant so that finally one has

$$V(x) = \frac{1}{2}w^2Q^2 = \frac{1}{2}w^2\left(\int^x \sqrt{M(s)}\,\mathrm{d}s + \alpha\right)^2.$$

Inserting the expression for $V(x)$ on the left-hand side from (2.19), we get

$$\int^x M(s)q_0(s)p_1(s)\,\mathrm{d}s = \frac{1}{2}w^2\left(\int^x \sqrt{M(s)}\,\mathrm{d}s + \alpha\right)^2. \tag{3.2}$$

Similarly in the case of a canonical transformation to the isotonic potential it is necessary that

$$\int^x M(s)q_0(s)p_1(s)\,\mathrm{d}s = \frac{1}{8}w^2\left(\int^x \sqrt{M(s)}\,\mathrm{d}s + \alpha\right)^2 + \frac{c^2}{\left(\int^x \sqrt{M(s)}\,\mathrm{d}s + \alpha\right)^2}. \tag{3.3}$$

Therefore the problem of identifying isochronous cases reduces to finding the explicit forms of p_1 and q_0, q_2 which ensure that the conditions stated in (3.2) or (3.3) respectively be fulfilled. The method adopted by Hill *et al* in [15] begins by recasting the original system of equations (2.2) into a complex form and seeks a direct transformation which reduces the system to the form $\dot{x} = y$ and $\dot{y} = -x$. In deriving such a transformation they make use of Pleshkan polynomials Π_j $(j \geqslant 1)$, the vanishing of which provides the necessary and sufficient conditions for the origin to be an isochronous center. The algorithmic nature of the calculation follows from a use of the Hilbert basis theorem. By using the symbolic computation package (REDUCE) they deduced the values of the coefficients of the polynomials p_i $(i = 0, 1)$ and q_j $(j = 0, 1, 2)$ such that the origin is an isochronous center.

Our method is more direct and is able to reproduce all the relevant results in [15]. We illustrate the method by considering a special case of the Cherkas system (2.1), (when $a_4 = 0$), below.

4. Isochronous cases of the reduced Cherkas system

For the Cherkas system (2.1) the polynomials have the following forms:

$$p_1(x) = (1+x), \tag{4.1}$$

$$q_0(x) = -(x + a_1x^2 + a_2x^3 + a_3x^4), \qquad q_2(x) = -a_6. \tag{4.2}$$

Consequently from (2.4) we have

$$f(x) = \frac{(a_6-1)}{1+x}, \qquad g(x) = (1+x)(x + a_1x^2 + a_2x^3 + a_3x^4). \tag{4.3}$$

The JLM from (2.12) is given by

$$M(x) = (1+x)^{2(a_6-1)}, \tag{4.4}$$

and the Lagrangian from (2.16) is

$$L(x,\dot{x}) = (1+x)^{2(a_6-1)}\frac{1}{2}\dot{x}^2 + \int^x (1+s)^{2a_6-1}q_0(s)\,\mathrm{d}s. \tag{4.5}$$

Expressing $q_0(s)$ as a polynomial is $(1+s)$ we have

$$L(x,\dot{x}) = (1+x)^{2(a_6-1)}\frac{\dot{x}^2}{2} - \int^x \sum_{k=0}^{4}\mu_k(1+s)^{2a_6+k-1}\,\mathrm{d}s, \tag{4.6}$$

where the $\mu_k,\ k = 0,\ldots,4$, have the following values:

$$\mu_0 = a_1 - a_2 + a_3 - 1, \qquad \mu_1 = -2a_1 + 3a_2 - 4a_3 + 1, \tag{4.7}$$

$$\mu_2 = a_1 - 3a_2 + 6a_3, \qquad \mu_3 = a_2 - 4a_3, \qquad \mu_4 = a_3. \tag{4.8}$$

The corresponding expression for the Hamiltonian is

$$H = \frac{1}{2}\left((1+x)^{1-a_6}p\right)^2 + \int^x \sum_{k=0}^{4}\mu_k(1+s)^{2a_6+k-1}\,\mathrm{d}s. \tag{4.9}$$

It is now evident that the potential is given by

$$V(x) = \sum_{k=0}^{4}\mu_k\int^x (1+s)^{2a_6+k-1}\,\mathrm{d}s = \sum_{k=0}^{4}\mu_k\frac{(1+x)^{2a_6+k}}{2a_6+k} \qquad (a_6 \neq 0). \tag{4.10}$$

If the values of the coefficient a_6 are such that $2a_6 + k = 0$, then it is to be assumed that the corresponding μ_k of the numerator vanishes.

4.1. *Case I: V(x) is mapped to the linear harmonic oscillator potential*

We look for a transformation such that

$$V(x) = \frac{1}{2}w^2\left(\frac{(1+x)^{a_6}}{a_6} + \alpha\right)^2 + \gamma, \tag{4.11}$$

where α and γ are constants. Here we have included γ to account for a possible shift. From (4.10) we have

$$\frac{w^2}{2a_6^2}(1+x)^{2a_6} + \frac{w^2\alpha}{a_6}(1+x)^{a_6} + \gamma + \frac{1}{2}w^2\alpha^2 = \sum_{k=0}^{4}\frac{\mu_k}{2a_6+k}(1+x)^{2a_6+k}. \tag{4.12}$$

Note that the right-hand side of (4.12) consists of five terms while the left-hand side has only three. Equating the coefficients of $(1+x)^{2a_6}$ we find that

$$w^2 = \mu_0 a_6. \tag{4.13}$$

Table 1. List of cases when $V(x)$ is mapped to a linear harmonic oscillator potential.

Case I Subcase	Parameter values	Potential $V(x)$	Canonical variables	Cf [15] theorem 7
(i) $a_6 = -1$	$a_1 = a_2 = a_3 = 0$	$\frac{1}{2}\left(1-(1+x)^{-1}\right)^2 -\frac{1}{2}$	$P = p(1+x)^2$ $Q = 1-(1+x)^{-1}$	Case 2
(iia) $a_6 = -2$	$a_1 = \frac{1}{2}, a_2 = a_3 = 0$	$\frac{1}{2}\left(\frac{1}{2}-\frac{1}{2}(1+x)^{-2}\right)^2 -\frac{1}{8}$	$P = p(1+x)^3$ $Q = \frac{1}{2}-\frac{1}{2}(1+x)^{-2}$	Case 4
(iiia) $a_6 = -3$	$a_1 = 1, a_2 = \frac{1}{3}, a_3 = 0$	$\frac{1}{2}\left(\frac{1}{3}-\frac{1}{3}(1+x)^{-3}\right)^2 -\frac{1}{18}$	$P = p(1+x)^4$ $Q = \frac{1}{3}-\frac{1}{3}(1+x)^{-3}$	Case 6
(iv) $a_6 = -4$	$a_1 = \frac{3}{2}, a_2 = \frac{1}{2}, a_3 = \frac{1}{4}$	$\frac{1}{2}\left(\frac{1}{4}-\frac{1}{4}(1+x)^{-4}\right)^2 -\frac{1}{32}$	$P = p(1+x)^5$ $Q = \frac{1}{4}-\frac{1}{4}(1+x)^{-4}$	Case 7

Table 2. An alternative case where $V(x)$ is mapped to a linear harmonic oscillator with $a_b = -2$.

Case I Subcase	Parameter values	Potential $V(x)$	Transformation variables	Cf [15] theorem 7
(iib) $a_6 = -2$	$a_1 = 2, a_2 = 1, a_3 = 0$	$\frac{1}{2}(1-(1+x)^{-1})^2 -\frac{1}{2}$	$P = p(1+x)^3$ $Q = 1-(1+x)^{-1}$	Case 5

The absence of a $(1 + x)$ independent term obviously implies $\gamma + 1/2w^2\alpha^2 = 0$. We can identify four possible subcases wherein either

(i) $a_6 = 2a_6 + 1$ with $\mu_1 \neq 0$ and $\mu_2 = \mu_3 = \mu_4 = 0$, s.t. $w^2\alpha = \mu_1 a_6/(2a_6 + 1)$,
(ii) $a_6 = 2a_6 + 2$ with $\mu_2 \neq 0$ and $\mu_1 = \mu_3 = \mu_4 = 0$, s.t. $w^2\alpha = \mu_2 a_6/(2a_6 + 2)$,
(iii) $a_6 = 2a_6 + 3$ with $\mu_3 \neq 0$ and $\mu_1 = \mu_2 = \mu_4 = 0$, s.t. $w^2\alpha = \mu_3 a_6/(2a_6 + 3)$,
(iv) $a_6 = 2a_6 + 4$ with $\mu_4 \neq 0$ and $\mu_1 = \mu_2 = \mu_3 = 0$, s.t. $w^2\alpha = \mu_4 a_6/(2a_6 + 4)$.

In subcase (i) $a_6 = -1$ while $\mu_2 = \mu_3 = \mu_4 = 0$ imply $a_1 = a_2 = a_3 = 0$. Using these values, it turns out that $\mu_1 = 1$ and $\mu_0 = -1$ from (4.7) while from (4.13) it follows that $w^2 = 1$. Hence $\alpha = a_6/(2a_6 + 1) = 1$ and we find that $\gamma = -1/2$. Therefore, the potential may be written as

$$V(x) = \tfrac{1}{2}(1-(1+x)^{-1})^2 - \tfrac{1}{2}.$$

The remaining subcases can be handled similarly. We present the results in table 1.

There is an additional case hidden in subcase (iia) which corresponds to the choice $\mu_0 = \mu_1 = 0$. In this case, one can still write the potential $V(x)$ as a perfect square, as shown in table 2. Figure 1 depicts the graphs of the potential $V(x)$ occurring in column 3 of tables 1 and 2, respectively.

4.2. Case II: when V(x) is mapped to the isotonic potential

As mentioned earlier, we attempt to rewrite the potential $V(x)$ of (4.10) as

$$\sum_{k=0}^{4} \mu_k \frac{(1+x)^{2a_6+k}}{2a_6+k} = \frac{1}{8}w^2\left(\frac{(1+x)^{a_6}}{a_6}+\alpha\right)^2 + \frac{c^2}{\left(\frac{(1+x)^{a_6}}{a_6}+\alpha\right)^2}.$$

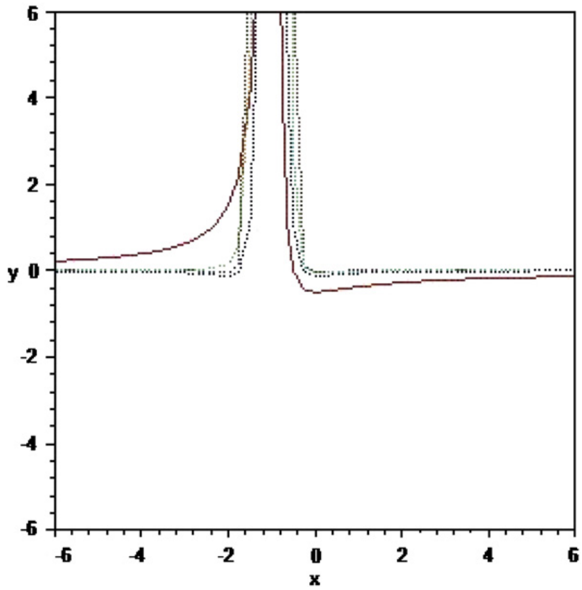

Figure 1. Graphs of the potentials $V(x)$ in table 1.

We have found that one can set $\alpha = 0$ so that

$$\sum_{k=0}^{4} \frac{\mu_k}{a_6^2(2a_6+k)}(1+x)^{4a_6+k} = \frac{1}{8}w^2\left(\frac{(1+x)^{4a_6}}{a_6^4}\right) + c^2.$$

The equation of the coefficients of $(1+x)^{4a_6}$ gives

$$w^2 = 4\mu_0 a_6. \tag{4.14}$$

We can then identify four subcases which are enumerated below.

(i) $4a_6 + 1 = 0$ with $\mu_1 = a_6^2(2a_6+1)c^2$ and $\mu_2 = \mu_3 = \mu_4 = 0$,
(ii) $4a_6 + 2 = 0$ with $\mu_2 = a_6^2(2a_6+2)c^2$ and $\mu_1 = \mu_3 = \mu_4 = 0$,
(iii) $4a_6 + 3 = 0$ with $\mu_3 = a_6^2(2a_6+3)c^2$ and $\mu_1 = \mu_2 = \mu_4 = 0$,
(iv) $4a_6 + 4 = 0$ with $\mu_4 = a_6^2(2a_6+4)c^2$ and $\mu_1 = \mu_2 = \mu_3 = 0$.

The unknown coefficients, a_i, are determined from the vanishing of the respective μ_k as listed above and subsequently one may determine μ_0 and the nonvanishing μ_k in each case. The latter fixes the value of the nonzero constant c^2. The results are presented in table 3.

In all the above results it is assumed that $(1+x) > 0$. Also we have not simplified the potentials any further to enable easier identification with the isotonic potential. The graphs of the potential $V(x)$ occurring in table 3 are shown in figure 2.

5. A cubic system

In [14] the author has analyzed the following five-parameter system:

$$\dot{x} = -y + bx^2y, \; \dot{y} = x + a_1x^2 + a_3y^2 + a_4x^3 + a_6xy^2, \tag{5.1}$$

under the condition $a_1 \neq a_3$, and has identified two new isochronous cases which are not included in the previous classification by Chavarriga and Garcia [11, 12]. We show how the same may be carried out in a simple way following roughly the method outlined above.

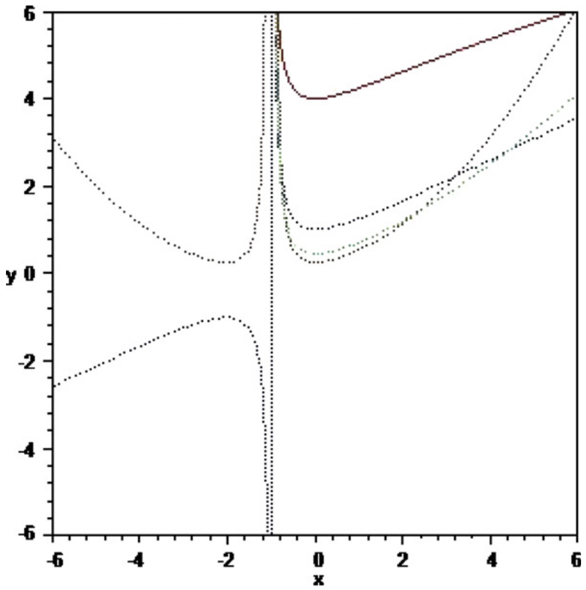

Figure 2. Graphs of the potentials $V(x)$ in table 3.

Table 3. List of cases when $V(x)$ is mapped to an isotonic potential.

Case II Subcase	Parameter values	Potential $V(x)$	Canonical variables	Cf [15] theorem 7
(i) $a_6 = -\frac{1}{4}$	$a_1 = a_2 = a_3 = 0$	$\frac{1}{8}(-4(1+x)^{-1/4})^2 + \frac{32}{(-4(1+x)^{-1/4})^2}$	$P = p(1+x)^{5/4}$ $Q = -4(1+x)^{-1/4}$	Case 3
(ii) $a_6 = -\frac{1}{2}$	$a_1 = \frac{1}{2}, a_2 = a_3 = 0$	$\frac{1}{8}(-2(1+x)^{-1/2})^2 + \frac{2}{(-2(1+x)^{-1/2})^2}$	$P = p(1+x)^{3/2}$ $Q = -2(1+x)^{-1/2}$	Case 1
(iii) $a_6 = -\frac{3}{4}$	$a_1 = 1, a_2 = \frac{1}{3}, a_3 = 0$	$\frac{1}{8}\left(-\frac{4}{3}(1+x)^{-3/4}\right)^2 + \frac{32/81}{\left(-\frac{4}{3}(1+x)^{-3/4}\right)^2}$	$P = p(1+x)^{7/4}$ $Q = -\frac{4}{3}(1+x)^{-3/4}$	Case 9
(iv) $a_6 = -1$	$a_1 = \frac{3}{2}, a_2 = 1, a_3 = \frac{1}{4}$	$\frac{1}{8}(-(1+x)^{-1})^2 + \frac{1/8}{(-(1+x)^{-1})^2}$	$P = p(1+x)^2$ $Q = -(1+x)^{-1}$	Case 8

The system (5.1) is equivalent to the following differential equation:

$$\ddot{x} = \frac{(a_3 + (a_6 + 2b)x)}{bx^2 - 1}\dot{x}^2 + (bx^2 - 1)(x + a_1x^2 + a_4x^3). \tag{5.2}$$

The corresponding equation for the JLM is

$$\frac{\mathrm{d}}{\mathrm{d}t}\log M + \frac{(a_3 + (a_6 + 2b)x)}{bx^2 - 1}2\dot{x} = 0,$$

and has the following solution:

$$M(x) = (bx^2 - 1)^{-(2b+a_6)/b}\left(\frac{1+\sqrt{b}x}{1-\sqrt{b}x}\right)^{\frac{a_3}{\sqrt{b}}}. \tag{5.3}$$

Then from $M = \partial^2 L/\partial \dot{x}^2$, it follows that an appropriate Lagrangian for (5.2) is given by

$$L = \tfrac{1}{2} M(x)\dot{x}^2 - V(x), \tag{5.4}$$

where the 'potential' $V(x)$ satisfies

$$V'(x) = -(bx^2 - 1)^{-(1+a_6/b)} \left(\frac{1+\sqrt{b}x}{1-\sqrt{b}x} \right)^{\frac{a_3}{\sqrt{b}}} (x + a_1 x^2 + a_4 x^3). \tag{5.5}$$

The above form of the Lagrangian is suggestive of the fact that the *JLM*, $M(x)$, *may be considered as playing the role of the mass (variable) of a particle undergoing one-dimensional motion*. The corresponding Hamiltonian may be written as

$$H = \frac{1}{2}\left(\frac{p}{\sqrt{M}} \right)^2 + V(x) \tag{5.6}$$

where the conjugate momenta $p = \partial L/\partial \dot{x} = M(x)\dot{x}$. Defining

$$P = \frac{p}{\sqrt{M}} \qquad \text{and} \qquad Q = Q(x) \tag{5.7}$$

to be some function of x such that the Poisson bracket $\{P, Q\} = \{p, x\}$ is invariant, we are led to conclude that $Q'(x) = \sqrt{M(x)}$. Suppose there exists a linearizing transformation such that $V(x) \longrightarrow Q^2/2$, then $V'(x) = QQ'(x) = \sqrt{M(x)}Q$, so that

$$Q = \frac{V'(x)}{\sqrt{M(x)}} = -(bx^2 - 1)^{-\frac{a_6}{2b}} (1 - \sqrt{b}x)^{-\frac{a_3}{2b}} (1 + \sqrt{b}x)^{\frac{a_3}{2b}} (x + a_1 x^2 + a_4 x^3). \tag{5.8}$$

Differentiating (5.8) w.r.t. x and as $Q'(x) = \sqrt{M(x)}$ we arrive at the following identity, after using (5.3):

$$a_4(a_6 - 3b)x^4 + (a_4 a_3 + a_1 a_6 - 2a_1 b)x^3 + (a_1 a_3 + a_6 + 3a_4 - b)x^2 + (a_3 + 2a_1)x = 0,$$

which implies

$$a_4(a_6 - 3b) = 0, \tag{5.9}$$

$$a_4 a_3 + a_1(a_6 - 2b) = 0, \tag{5.10}$$

$$a_1 a_3 + a_6 + 3a_4 - b = 0, \tag{5.11}$$

$$a_3 + 2a_1 = 0. \tag{5.12}$$

Assuming $a_1 \neq a_3$ one can see that the above equations admit the following sets of solutions:

$$(S_1): \quad a_1 = -\frac{a_3}{2}, \qquad a_4 = 0, \qquad a_6 = a_3^2, \qquad b = \frac{a_3^2}{2} \tag{5.13}$$

$$(S_2): \quad a_1 = -\frac{a_3}{2}, \qquad a_4 = \frac{a_3^2}{14}, \qquad a_6 = \frac{3a_3^2}{7}, \qquad b = \frac{a_3^2}{7}. \tag{5.14}$$

These are precisely the values obtained by Chouikha in [14]. For the case S_1 the JLM has the following explicit expression:

$$M(x) = \left(1 + \frac{a_3 x}{\sqrt{2}} \right)^{-4+\sqrt{2}} \left(1 - \frac{a_3 x}{\sqrt{2}} \right)^{-(4+\sqrt{2})}, \tag{5.15}$$

while the coordinate Q is given by

$$Q(x) = \frac{x(a_3 x - 2)(2 - \sqrt{2}a_3 x)^{-\frac{1}{\sqrt{2}}} (2 + \sqrt{2}a_3 x)^{\frac{1}{\sqrt{2}}}}{\left(a_3^2 x^2 - 2\right)}. \tag{5.16}$$

One can integrate (5.6) to obtain the potential $V(x)$ which in this case appears as

$$V(x) = \frac{x^2(a_3x-2)^2(2-\sqrt{2}a_3x)^{-\sqrt{2}}(2+\sqrt{2}a_3x)^{\sqrt{2}}}{2\left(a_3^2x^2-2\right)^2} = \frac{1}{2}Q^2. \tag{5.17}$$

Thus the potential can be mapped to that of a linear harmonic oscillator and we conclude that the origin is an isochronous center.

The corresponding expression for the JLM $M(x)$, the coordinate Q and the potential $V(x)$, in case S_2, are as follows:

$$M(x) = \left(1+\frac{a_3x}{\sqrt{7}}\right)^{-5+\sqrt{7}}\left(1-\frac{a_3x}{\sqrt{7}}\right)^{-5-\sqrt{7}}, \tag{5.18}$$

$$Q(x) = \frac{x\left(1-\frac{a_3x}{\sqrt{7}}\right)^{-\frac{\sqrt{7}}{2}}\left(1+\frac{a_3x}{\sqrt{7}}\right)^{\frac{\sqrt{7}}{2}}\left(a_3^2x^2-7a_3x+14\right)}{14\left(1-\frac{a_3^2x^2}{7}\right)^{3/2}}, \tag{5.19}$$

$$V(x) = \frac{x^2\left(1-\frac{a_3x}{\sqrt{7}}\right)^{-\sqrt{7}}\left(1+\frac{a_3x}{\sqrt{7}}\right)^{\sqrt{7}}\left(a_3^2x^2-7a_3x+14\right)^2}{392\left(1-\frac{a_3^2x^2}{7}\right)^{3}} = \frac{1}{2}Q^2. \tag{5.20}$$

From the expressions for M one can easily construct P, using (5.7),

$$P = \sqrt{M}\dot{x} = \sqrt{M}(bx^2-1)y, \tag{5.21}$$

and the appropriate values of b for cases S_1 and S_2, respectively.

6. Conclusion and outlook

The study of the isochronicity problem for parametric families of ODEs consists of a difficult computational problem to find the necessary conditions for isochronicity. So it is desirable to find a new approach to this problem. The manner in which we have identified the isochronous cases of the Cherkas system rests entirely on theorem 1 of [10]. It does not require any numerical computation. However, we would like to point out that the paper of Hill *et al* [15] mentions a tenth case under theorem 7. We have not been able to include this particular case in the above formalism, since it requires $a_4 \neq 0$. Furthermore, the procedure adopted in this paper, based on the properties of the Jacobi last multiplier, appears to be applicable to many other systems as shown by the analysis of the cubic system studied in [14] and has the advantage of being remarkably simple.

Finally, it is known that apart from the harmonic or isotonic oscillators, there exist a large class of isochronous potentials which are all nonpolynomial and nonsymmetric [23]. An interesting open question naturally arises in this connection, whether there are polynomial systems of ODEs mapping to other classes of isochronous potentials. In this paper, we have demonstrated that mapping to various other isochronous potentials is a very efficient method to check isochronicity of some parametric families of systems of ODEs.

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